Final Exam - Optimization
B. Math III

25 April, 2025

(i) Duration of the exam is 3 hours.

(ii) The maximum number of points you can score in the exam is 100 (total = 105).

(iii) You are not allowed to consult any notes or external sources for the exam.

Name: ______________________________

Roll Number: ______________________________

1. (20 points) Prove that a polytope in $\mathbb{R}^n$ has finitely many extreme points. (A polyhedron is defined by finitely many linear equalities and inequalities. A polytope is a bounded polyhedron.)

Total for Question 1: 20

2. Consider a linear programming problem in standard form, described in terms of the following initial tableau:

0	0	0	0	δ	3	γ	ξ
β	0	1	0	α	1	0	3
2	0	0	1	-2	2	η	-1
3	1	0	0	0	-1	2	1

The entries $\alpha, \beta, \gamma, \delta, \eta, \xi$ in the tableau are unknown parameters. Furthermore, let $\mathcal{B}$ be the basis matrix corresponding to having x_2, x_3, and x_1 (in that order) be the basic variables. For each one of the following statements, with justification find the ranges of values of the various parameters that will make the statement to be true. (No marks will be given if justification of answer is not provided.)

(a) (5 points) The first row in the present tableau (below the row with the reduced costs) indicates that the problem is infeasible.

(b) (5 points) The corresponding basic solution is feasible, but we do not have an optimal basis.

(c) (5 points) The corresponding basic solution is feasible, x_6 is a candidate for entering the basis, and when x_6 is the entering variable, x_3 leaves the basis.

(d) (5 points) The corresponding basic solution is feasible, x_7 is a candidate for entering the basis, but if it does, the solution and the objective value remain unchanged.

Total for Question 2: 20

3. While solving a linear programming problem by the simplex method, the following tableau is obtained at some iteration.

	0	$\dots$	0	$\bar{c}_{m+1}$	$\dots$	$\bar{c}_n$
x_1	1	$\dots$	0	$a_{1,m+1}$	$\dots$	$a_{1,n}$
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
x_m	0	$\dots$	1	$a_{m,m+1}$	$\dots$	$a_{m,n}$

Assume that in this tableau we have $\bar{c}_j \geq 0$ for $j = m+1, \dots, n-1$, and $\bar{c}_n < 0$. In particular, x_n is the only candidate for entering the basis.

(a) (10 points) Suppose that x_n indeed enters the basis and that this is a nondegenerate pivot (that is, $\theta^* \neq 0$). Prove that x_n will remain basic in all subsequent iterations of the algorithm and that x_n is a basic variable in any optimal basis.

(b) (5 points) Suppose that x_n indeed enters the basis and that this is a degenerate pivot (that is, $\theta^* = 0$). Show that x_n need not be basic in an optimal basic feasible solution.

Total for Question 3: 15

4. Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is convex and twice continuously differentiable. Suppose $\bar{y}$ and $\bar{x}$ are related by $\bar{y} = \nabla f(\bar{x})$, and that $\nabla^2 f(\bar{x})$ is positive-definite. Let f^* denote the Fenchel conjugate of f.

(a) (5 points) Show that $\nabla f^*(\bar{y}) = \bar{x}$.

(b) (10 points) Show that $\nabla^2 f^*(\bar{y}) = \nabla^2 f(\bar{x})^{-1}$.

Total for Question 4: 15

5. For the convex optimization problem below,

$$\begin{array}{ll} \text{minimize} & x^2 + 1 \\ \text{subject to} & (x-2)^2 \leq 1, \quad x \in \mathbb{R} \end{array}$$

do the following.

(a) (3 points) Formulate the Lagrangian function.

(b) (3 points) Derive the Lagrangian dual function.

(c) (4 points) Verify that strong duality holds.

Total for Question 5: 10

6. (10 points) Solve the following optimization problem:

$$\min_{x_1,x_2} -2\left(x_1-2\right)^2 - x_2^2$$
$$\text{subject to } x_1^2 + x_2^2 \leq 25$$
$$x_1 \geq 0.$$

Total for Question 6: 10

7. A portfolio manager wants to minimize risk while achieving a required return by investing a total budget B across 2 assets. The problem is defined as:

$$\min_{\vec{x}\in\mathbb{R}^2} \frac{1}{2}\vec{x}^T Q\vec{x},$$

subject to:

$$\vec{r}^T\vec{x} \geq R, \quad \sum_{i=1}^{n} x_i = B, \quad x_i \geq 0 \quad \forall i.$$

- Q: The covariance matrix quantifying asset risk and correlations. Diagonal elements (Q_{ii}) measure individual asset risk, while off-diagonal elements (Q_{ij}) capture inter-asset risk.
- $\vec{r}$: The expected return vector, where each r_i represents the return per unit investment in asset i.
- $\vec{x}$: The allocation vector, where x_i is the amount invested in asset i.
- R: The minimum required portfolio return.
- B: The total budget to be allocated across all assets.

(a) (10 points) Solve for the optimal allocations $\mathbf{x}^*$ and the optimal dual variables when:

$$Q = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}, \quad r = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad R = 0.15, \quad B = 1.$$

(b) (5 points) Interpret the dual variable λ^* as the sensitivity of the optimal risk to changes in R. Estimate the change in risk if R increases by $\Delta R = 0.05$.

Total for Question 7: 15